Formulas for calculating the extremal ranks and inertias of a matrix-valued function subject to matrix equation restrictions

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Abstract. Matrix rank and inertia optimization problems are a class of discontinuous optimization problems in which the decision variables are matrices running over certain matrix sets, while the ranks and inertias of the variable matrices are taken as integer-valued objective functions. In this paper, we establish a group of explicit formulas for calculating the maximal and minimal values of the rank and inertia objective functions of the Hermitian matrix expression $A_1 - B_1XB_1^*$ subject to the common Hermitian solution of a pair of consistent matrix equations $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$, and Hermitian solution of the consistent matrix equation $B_4X = A_4$, respectively. Many consequences are obtained, in particular, necessary and sufficient conditions are established for the triple matrix equations $B_1XB_1^* = A_1$, $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ to have a common Hermitian solution, as necessary and sufficient conditions for the two matrix equations $B_1XB_1^* = A_1$ and $B_4X = A_4$ to have a common Hermitian solution.

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1 Introduction

Throughout this paper,

$\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices;
$\mathbb{C}^{m}_{\mathrm{H}}$ stands for the set of all $m \times m$ complex Hermitian matrices;

A^T, A^*, $r(A)$, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ stand for the transpose, conjugate transpose, rank, range (column space) and null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively;

I_m denotes the identity matrix of order m;

$[\,A,\, B\,]$ denotes a row block matrix consisting of A and B;

$A > 0$ ($A \geqslant 0$) means that A is Hermitian positive definite (Hermitian positive semi-definite);

two $A,\, B \in \mathbb{C}^{m}_{\mathrm{H}}$ are said to satisfy the inequality $A > B$ ($A \geqslant B$) in the Löwner partial ordering if $A - B$ is positive definite (positive semi-definite);

the Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution X satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$, which satisfies $AA^{\dagger} = A^{\dagger}A$ if $A = A^*$;

a matrix X is called a Hermitian g-inverse of $A \in \mathbb{C}^{m}_{\mathrm{H}}$, denoted by A^{-}, if it satisfies both $AXA = A$ and $X = X^*$;

E_A and F_A stand for $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$. The ranks of E_A and F_A are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$;

$i_{+}(A)$ and $i_{-}(A)$, usually called the partial inertia of $A \in \mathbb{C}^{m}_{\mathrm{H}}$, are defined to be the numbers of the positive and negative eigenvalues of A counted with multiplicities, respectively, which satisfy $r(A) = i_{+}(A) + i_{-}(A)$.

The matrix approximation problem is to approximate optimally, with respect to some criteria, a matrix by one of the same dimension from a given feasible matrix set. Assume that A is a matrix to be approximated. Then a conventional statement of general matrix optimization problems of A from this point of view can be written as

$$\text{minimize } \rho(\,A - X\,) \ \text{ subject to } X \in \mathcal{S}, \tag{1.1}$$

where $\rho(\cdot)$ is certain objective function, which is usually taken as the determinant, trace, norms, rank, inertia of matrix, and $\mathcal{S}$ is a given feasible matrix set. A best-known case of (1.1) is to minimize the norm $\|\, A - X\,\|_F^2$ subject to $X \in \mathcal{S}$.

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In this paper, we take the matrix set $\mathcal{S}$ as

$$\mathcal{S} = \{\, \phi(X) = A_1 - B_1XB_1^* \mid [\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,] \,\}, \tag{1.2}$$

$$\mathcal{S} = \{\, \phi(X) = A_1 - B_1XB_1^* \mid B_4X = A_4 \,\}, \tag{1.3}$$

where $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$, $A_4,\ B_4 \in \mathbb{C}^{m\times n}$ are given, $i = 1, 2, 3$, and $X \in \mathbb{C}_{\mathrm{H}}^{n}$ is a variable matrix, and study the following constrained optimization problems:

Problem 1.1 For the constrained linear matrix-valued function in (1.2), establish explicit formulas for calculating

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } [\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,], \tag{1.4}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } [\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,], \tag{1.5}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } [\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,], \tag{1.6}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } [\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,]. \tag{1.7}$$

Problem 1.2 For the constrained linear matrix-valued function in (1.2),

(i) establish necessary and sufficient conditions for the following three matrix equations

$$[\, B_1XB_1^*,\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_1,\, A_2,\, A_3 \,] \tag{1.8}$$

to have a common Hermitian solution;

(ii) establish necessary and sufficient conditions for $A_1 - B_1XB_1^* > (\geqslant, <, \leqslant)\, 0$ to hold for an $X \in \mathbb{C}_{\mathrm{H}}^{n}$ satisfying $[\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,]$;

(iii) establish necessary and sufficient conditions for $A_1 - B_1XB_1^* > (\geqslant, <, \leqslant)\, 0$ to hold for all $X \in \mathbb{C}_{\mathrm{H}}^{n}$ satisfying $[\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,]$, namely, $A_1 - B_1XB_1^*$ is a positive map under the restriction $[\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,]$.

Problem 1.3 For the constrained linear matrix-valued function in (1.3), establish explicit formulas for calculating

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } B_4X = A_4, \tag{1.9}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } B_4X = A_4, \tag{1.10}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } B_4X = A_4, \tag{1.11}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(\, A_1 - B_1XB_1^* \,) \qquad \text{s.t. } B_4X = A_4. \tag{1.12}$$

Problem 1.4 For the linear matrix map in (1.3),

(i) establish necessary and sufficient conditions for the following two matrix equations $[\, B_1XB_1^*,\, B_4X \,] = [\, A_1,\, A_4 \,]$ to have a common Hermitian solution and nonnegative definite solution;

(ii) establish necessary and sufficient conditions for $A_1 - B_1XB_1^* > (\geqslant, <, \leqslant)\, 0$ to hold for an $X \in \mathbb{C}_{\mathrm{H}}^{n}$ satisfying $B_4X = A_4$;

(iii) establish necessary and sufficient conditions for $A_1 - B_1XB_1^* > (\geqslant, <, \leqslant)\, 0$ to hold for all $X \in \mathbb{C}_{\mathrm{H}}^{n}$ satisfying $B_4X = A_4$, namely, $A_1 - B_1XB_1^*$ is a positive map under the restriction $B_4X = A_4$.

The extremal ranks and inertias of a matrix expression can directly be used to describe some behaviors of the matrix expression, for example,
(I) the maximal and minimal dimensions of the row and column spaces of the matrix expression;
(II) nonsingularity of the matrix expression when it is square;
(III) solvability of the corresponding matrix equation;
(IV) rank, inertia and range invariance of the matrix expression;
(V) semi-definiteness of a matrix expression; etc.
On the other hand, matrix rank and inertia optimization problems are NP-hard in general due to the discontinuity and combinational nature of rank and inertia of a matrix and the complexity of algebraic structure of $\mathcal{S}$.

Mappings between matrix spaces with symmetric patterns can be constructed arbitrarily, but the linear map $\phi(X)$ in (1.2) and (1.3) is the simplest cases among all LMFs with symmetric patterns. This matrix-valued function is the starting point in dealing with various complicated matrix-valued functions with symmetric patterns. In recent years, the matrix-valued function $\phi(X) = A - BXB^*$ was reconsidered and many new results on its algebraic properties were obtained, for instance,

(i) Expansion formulas for calculating the (global extremal) rank and inertia of $\phi(X)$ when X running over $\mathbb{C}_{\rm H}^{n}$, see [19, 30, 38].

(ii) Nonsingularity, positive definiteness, rank and inertia invariance, etc., of the $\phi(X)$, see [30, 38].

(iii) Canonical forms of the $\phi(X)$ under generalized singular value decompositions and their algebraic properties, see [19].

(iv) Solutions and least-squares solutions of the matrix equation $\phi(X) = 0$ and their algebraic properties, see [16, 20, 33, 36, 37].

(v) Solutions of the matrix inequalities $\phi(X) > (\geqslant, <, \leqslant)\, 0$ and their properties, see [30].

(vi) Minimization of $\mathrm{tr}[\,\phi(X)\phi^*(X)\,]$ s.t. $r[\phi(X)] = \min$, see [37].

(vii) Formulas for calculating the extremal rank and inertia of $\phi(X)$ under the restrictions $r(X) \leqslant k$ and/or $\pm X \geqslant 0$, see [35].

(viii) Formulas for calculating the extremal rank and inertia of $\phi(X)$ subject to a consistent matrix equation $CXC^* = D$, see [18].

This basic work was also extended to some general LMFs, such as, $A - BX - (BX)^*$, $A - BXB^* - CYC^*$ and $A - BXC - (BXC)^*$, where X and Y are (Hermitian) variable matrices of appropriate sizes; see [2, 15, 16, 17, 18, 32, 33, 36].

We shall use some pure algebraic operations on matrices to derive two groups of analytical formulas for calculating the global extremal values of the objective functions in (1.4)–(1.7) and (1.9)–(1.12), and then to present a variety of valuable consequences of these formulas.

Since variable entries in a matrix-valued function are often regarded as continuous variables in some constrained sets, while the objective functions—the rank and inertia of the matrix-valued function take values only from a finite set of nonnegative integers, Hence, (1.4)–(1.7) and (1.9)–(1.12) can be regarded as continuous-integer optimization problems subject to equality constraints. This kind of non-smooth optimization problems cannot be solved by using various optimization methods for solving continuous or discrete cases. There is no rigorous mathematical theory for solving a general rank and inertia optimization problem due to the discontinuity and nonconvexity of rank and inertia of matrix. In fact, it has been realized that rank and inertia optimization problems have deep connections with computational complexity, and are regarded as NP-hard in general; see, e.g., [1, 3, 4, 5, 7, 8, 9, 12, 22, 25, 27]. Fortunately, some special rank and inertia optimization problems now can be solved by pure algebraical methods. In particular, analytical solutions to the rank and inertia optimization problems of the $\phi(X)$ in (1.2) and (1.3), as well as (1.4)–(1.7) and (1.9)–(1.12) can be derived algebraically by using generalized inverses of matrices.

The following are some known results for ranks and inertias of matrices and their usefulness, which will be used in the latter part of this paper.

Lemma 1.5 ([30]) *Let $\mathcal{H}$ be a matrix set in $\mathbb{C}_{\rm H}^{m}$. Then,*

(a) *$\mathcal{H}$ has a matrix $X > 0$ $(X < 0)$ if and only if* $\max_{X\in\mathcal{H}} i_{+}(X) = m$ $\left(\max_{X\in\mathcal{H}} i_{-}(X) = m\right)$.

(b) *All $X \in \mathcal{H}$ satisfy $X > 0$ $(X < 0)$, namely, $\mathcal{H}$ is a subset of the cone of positive definite matrices (negative definite matrices), if and only if* $\min_{X\in\mathcal{H}} i_{+}(X) = m$ $\left(\min_{X\in\mathcal{H}} i_{-}(X) = m\right)$.

(c) *$\mathcal{H}$ has a matrix $X \geqslant 0$ $(X \leqslant 0)$ if and only if* $\min_{X\in\mathcal{H}} i_{-}(X) = 0$ $\left(\min_{X\in\mathcal{H}} i_{+}(X) = 0\right)$.

(d) *All $X \in \mathcal{H}$ satisfy $X \geqslant 0$ $(X \leqslant 0)$ namely, $\mathcal{H}$ is a subset of the cone of nonnegative definite matrices (semi-definite matrices), if and only if* $\max_{X\in\mathcal{H}} i_{-}(X) = 0$ $\left(\max_{X\in\mathcal{H}} i_{+}(X) = 0\right)$.

The question of whether a given function is (definite or semi-definite everywhere is ubiquitous in mathematics and applications. Lemma 1.5(a)–(d) show that if some explicit formulas for calculating the global maximal and minimal inertias of a given Hermitian matrix-valued function are established, we can use them, as demonstrated in Sections below, to derive necessary and sufficient conditions for the Hermitian matrix-valued function to be definite or semi-definite.

Lemma 1.6 ([21]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$. Then, the following rank expansion formulas hold*

$$r[\,A,\,B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.13}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.14}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C). \tag{1.15}$$

Three useful rank expansion formulas derived from (1.15) are

$$r\begin{bmatrix} A & B & 0 \\ C & 0 & P \end{bmatrix} = r(P) + r\begin{bmatrix} A & B \\ E_PC & 0 \end{bmatrix}, \tag{1.16}$$

$$r\begin{bmatrix} A & B \\ C & 0 \\ 0 & Q \end{bmatrix} = r(Q) + r\begin{bmatrix} A & BF_Q \\ C & 0 \end{bmatrix}, \tag{1.17}$$

$$r\begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix} = r(P) + r(Q) + r\begin{bmatrix} A & BF_Q \\ E_PC & 0 \end{bmatrix}. \tag{1.18}$$

We shall use them in Section 2 to simplify ranks of block matrices involving E_P and F_Q.

Lemma 1.7 ([30]) *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}_{\mathrm{H}}^{n}$, and let*

$$U = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad V = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the following expansion formulas hold

$$i_{\pm}(U) = r(B) + i_{\pm}(E_BAE_B), \tag{1.19}$$

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ B^*E_A & D - B^*A^{\dagger}B \end{bmatrix}. \tag{1.20}$$

(a) *If $A \geqslant 0$, then*

$$i_{+}(U) = r[\,A,\,B\,], \quad i_{-}(U) = r(B), \quad r(U) = r[\,A,\,B\,] + r(B). \tag{1.21}$$

(b) *If $A \leqslant 0$, then*

$$i_{+}(U) = r(B), \quad i_{-}(U) = r[\,A,\,B\,], \quad r(U) = r[\,A,\,B\,] + r(B). \tag{1.22}$$

(c) *If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, then*

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}(\,D - B^*A^{\dagger}B\,), \quad r(V) = r(A) + r(\,D - B^*A^{\dagger}B\,). \tag{1.23}$$

(d) *If $\mathscr{R}(B) \cap \mathscr{R}(A) = \{0\}$ and $\mathscr{R}(B^*) \cap \mathscr{R}(D) = \{0\}$, then*

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}(D) + r(B), \quad r(V) = r(A) + 2r(B) + r(D). \tag{1.24}$$

Three general expansion formulas derived from (1.19) are

$$i_{\pm}\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P), \; r\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - 2r(P). \tag{1.25}$$

We shall use them to simplify the inertias of block Hermitian matrices that involve $F_P = I - P^{\dagger}P$.

Lemma 1.8 *Let $A_j \in \mathbb{C}^{m_j\times n}$, $B_j \in \mathbb{C}^{p\times q_j}$ and $C_j \in \mathbb{C}^{m_j\times q_j}$ be given, $j = 1, 2$. Then,*

(a) [26] *The pair of matrix equations*

$$A_1XB_1 = C_1 \;\; \text{and} \;\; A_2XB_2 = C_2 \tag{1.26}$$

have a common solution for $X \in \mathbb{C}^{n\times p}$ if and only if

$$\mathscr{R}(C_j) \subseteq \mathscr{R}(A_j), \quad \mathscr{R}(C_j^*) \subseteq \mathscr{R}(B_j^*), \quad r\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r[\,B_1,\,B_2\,], \quad j = 1,\,2. \tag{1.27}$$

(b) [29] *Under (1.45), the general common solution to (1.44) can be written in the following parametric form*

$$X = X_0 + F_A V_1 + V_2 E_B + F_{A_1} V_3 E_{B_2} + F_{A_2} V_4 E_{B_1}, \tag{1.28}$$

where $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $B = [\, B_1,\, B_2 \,]$, *and the four matrices* $V_1, \dots, V_4 \in \mathbb{C}^{n\times p}$ *are arbitrary.*

Lemma 1.9 *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}_{\mathrm{H}}^{m}$ *be given. Then,*

(a) [6, 10] *The matrix equation* $AXA^* = B$ *has a solution* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *or equivalently,* $AA^{\dagger}B = B$.

(b) [30] *Under* $AA^{\dagger}B = B$, *the general Hermitian solution of* $AXA^* = B$ *can be written in the following two forms*

$$X = A^{\dagger}B(A^{\dagger})^* + U - A^{\dagger}AUA^{\dagger}A, \tag{1.29}$$

$$X = A^{\dagger}B(A^{\dagger})^* + F_A V + V^* F_A, \tag{1.30}$$

where $U \in \mathbb{C}_{\mathrm{H}}^{n}$ *and* $V \in \mathbb{C}^{n\times n}$ *are arbitrary.*

More results on properties of solutions of $AXA^* = B$ can be found in [16, 20].

Lemma 1.10 ([10]) *Let* $A,\, B \in \mathbb{C}^{m\times n}$ *be given. Then,*

(a) *The matrix equation* $AX = B$ *has a Hermitian solution* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $AB^* = BA^*$. *In this case, the general Hermitian solution of* $AX = B$ *can be written as*

$$X = A^{\dagger}B + (A^{\dagger}B)^* - A^{\dagger}BA^{\dagger}A + F_A U F_A, \tag{1.31}$$

where $U \in \mathbb{C}_{\mathrm{H}}^{n}$ *is arbitrary.*

(b) *The matrix equation* $AX = B$ *has a solution* $0 \leqslant X \in \mathbb{C}_{\mathrm{H}}^{n}$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, $AB^* \geqslant 0$ *and* $r(AB^*) = r(B)$. *In this case, the general solution of* $AX = B$ *can be written as*

$$X = B^*(AB^*)^{\dagger}B + F_A U F_A, \tag{1.32}$$

where $0 \leqslant U \in \mathbb{C}_{\mathrm{H}}^{n}$ *is arbitrary.*

Lemma 1.11 *Let* $A \in \mathbb{C}_{\mathrm{H}}^{m}$ *and* $B \in \mathbb{C}^{m\times n}$ *be given. Then,*

(a) [30, 38] *The global maximal and minimal ranks and inertias of* $A - BXB^*$ *subject to* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *are given by*

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A - BXB^* \,) = r[\, A,\, B \,], \tag{1.33}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A - BXB^* \,) = 2r[\, A,\, B \,] - r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{1.34}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(\, A - BXB^* \,) = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{1.35}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(\, A - BXB^* \,) = r[\, A,\, B \,] - i_{\mp}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}. \tag{1.36}$$

(b) [35] *The global maximal and minimal ranks and inertias of* $A - BXB^*$ *subject to* $0 \leqslant X \in \mathbb{C}_{\mathrm{H}}^{n}$ *are given by*

$$\max_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A + BXB^* \,) = r[\, A,\, B \,], \quad \min_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A + BXB^* \,) = i_{+}(A) + r[\, A,\, B \,] - i_{+}(M), \tag{1.37}$$

$$\max_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{+}(\, A + BXB^* \,) = i_{+}(M), \quad \min_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{+}(\, A + BXB^* \,) = i_{+}(A), \tag{1.38}$$

$$\max_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{-}(\, A + BXB^* \,) = i_{-}(A), \quad \min_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{-}(\, A + BXB^* \,) = r[\, A,\, B \,] - i_{+}(M), \tag{1.39}$$

$$\max_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A - BXB^* \,) = r[\, A,\, B \,], \quad \min_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} r(\, A - BXB^* \,) = i_{-}(A) + r[\, A,\, B \,] - i_{-}(M), \tag{1.40}$$

$$\max_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{+}(\, A - BXB^* \,) = i_{+}(A), \quad \min_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{+}(\, A - BXB^* \,) = r[\, A,\, B \,] - i_{-}(M), \tag{1.41}$$

$$\max_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{-}(\, A - BXB^* \,) = i_{-}(M), \quad \min_{0\leqslant X\in\mathbb{C}_{\mathrm{H}}^{n}} i_{-}(\, A - BXB^* \,) = i_{-}(A). \tag{1.42}$$

Lemma 1.12 ([18]) *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times m}$ be given, and let*

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix}, \tag{1.43}$$

$$N = [\, A, \, B, \, C^* \,], \quad N_1 = \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}. \tag{1.44}$$

Then, the global maximal and minimal ranks and partial inertias of $A - BXC - (BXC)^*$ are given by

$$\max_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^* \,] = \min\{r(N), \quad r(N_1), \quad r(N_2)\}, \tag{1.45}$$

$$\min_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^* \,] = 2r(N) + \max\{\, s_1, \quad s_2, \quad s_3, \quad s_4 \,\}, \tag{1.46}$$

$$\max_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^* \,] = \min\{i_{\pm}(M_1), \quad i_{\pm}(M_2)\}, \tag{1.47}$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^* \,] = r(N) + \max\{\, i_{\pm}(M_1) - r(N_1), \quad i_{\pm}(M_2) - r(N_2) \,\}, \tag{1.48}$$

where

$$s_1 = r(M_1) - 2r(N_1), \quad s_2 = r(M_2) - 2r(N_2),$$

$$s_3 = i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2), \quad s_4 = i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2).$$

In particular, if $\mathscr{R}(C^*) \subseteq \mathscr{R}(B)$, then

$$\max_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^* \,] = \min\left\{ r[\, A, \, B \,], \quad r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \tag{1.49}$$

$$\min_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^* \,] = 2r[\, A, \, B \,] + r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.50}$$

$$\max_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^* \,] = i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix}, \tag{1.51}$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^* \,] = r[\, A, \, B \,] + i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.52}$$

The matrices X that satisfy (1.45)–(1.48) (namely, the global maximizers and minimizers of the objective rank and inertia functions) are not necessarily unique and their expressions were also given in [18] by using certain simultaneous decomposition of the three given matrices. Observe that the right-hand sides of (1.45)–(1.48) are represented in analytical forms of the ranks and inertias of the five given block matrices, we can easily use them to derive extremal ranks and inertias of some general linear and nonlinear matrix-valued functions. In these cases, combining the rank and inertia formulas obtained with the assertions in Lemma 1.1 may yield various conclusions on algebraic properties of linear and nonlinear matrix-valued functions.

2 The extremal ranks and inertias of $A - B_1XB_1^*$ subject to $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$

We first derive a parametric form for the general common Hermitian solution of the pair of matrix equations in (1.2).

Lemma 3.1 ([33]) *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$, $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 2, \, 3,$ and suppose that each of the two matrix equations*

$$B_2XB_2^* = A_2 \quad \text{and} \quad B_3XB_3^* = A_3 \tag{3.1}$$

has a solution, i.e., $\mathscr{R}(A_i) \subseteq \mathscr{R}(B_i)$ for $i = 2, \, 3$. Then,

(a) *The pair of matrix equations have a common Hermitian solution if and only if*

$$r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & -A_3 & B_3 \\ B_2^* & B_3^* & 0 \end{bmatrix} = 2r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix}. \tag{3.2}$$

(b) *Under (3.2), the general common Hermitian solution of the pair of equations can be written in the following parametric form*

$$X = X_0 + VF_B + F_BV^* + F_{B_2}UF_{B_3} + F_{B_3}U^*F_{B_2}, \tag{3.3}$$

where X_0 is a special Hermitian common solution to the pair of equations, $B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$, and $U, \, V \in \mathbb{C}^{n\times n}$ are arbitrary.

Substituting (3.3) into $A_1 - B_1XB_1^*$ gives

$$A_1 - B_1XB_1^* = A_1 - B_1X_0B_1^* - B_1VF_BB_1^* - B_1F_BV^*B_1^* - B_1F_{B_2}UF_{B_3}B_1^* - B_1F_{B_3}U^*F_{B_2}B_1^*, \tag{3.4}$$

which is a matrix-valued function involving two variable matrices V and U. Thus, the constrained matrix-valued function in (1.2) is equivalently converted to the unconstrained matrix-valued function in (3.4). To find the global maximal and minimal ranks and partial inertias of (3.4), we need the following result.

Lemma 3.2 *Let*

$$\phi(X_1,\, X_2) = A - B_1X_1C_1 - (B_1X_1C_1)^* - B_2X_2C_2 - (B_2X_2C_2)^*, \tag{3.5}$$

where $A \in \mathbb{C}_{\rm H}^{m}$, $B_i \in \mathbb{C}^{m\times p_i}$ *and* $C_i \in \mathbb{C}^{q_i\times m}$ *are given, and* $X_i \in \mathbb{C}^{p_i\times q_i}$ *are variable matrices for* $i = 1, 2$, *and assume that*

$$\mathscr{R}(B_2) \subseteq \mathscr{R}(B_1), \quad \mathscr{R}(C_1^*) \subseteq \mathscr{R}(B_1), \quad \mathscr{R}(C_2^*) \subseteq \mathscr{R}(B_1). \tag{3.6}$$

Also let

$$N = \begin{bmatrix} A & B_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} A & B_2 & C_1^* & C_2^* \\ B_2^* & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} A & B_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} A & B_2 & C_1^* \\ B_2^* & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & C_1^* & C_2^* \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix}.$$

Then, the global maximal and minimal ranks and partial inertias of $\phi(X_1,\, X_2)$ *are given by*

$$\max_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} r[\,\phi(X_1,\, X_2)\,] = \min\{r[\,A,\, B_1\,],\quad r(N),\quad r(M_1),\quad r(M_2)\}, \tag{3.7}$$

$$\min_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} r[\,\phi(X_1,\, X_2)\,] = 2r[\,A,\, B_1\,] - 2r(M) + 2r(N) + \max\{\, s_1,\quad s_2,\quad s_3,\quad s_4\,\}, \tag{3.8}$$

$$\max_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} i_{\pm}[\,\phi(X_1,\, X_2)\,] = \min\{i_{\pm}(M_1),\quad i_{\pm}(M_2)\}, \tag{3.9}$$

$$\begin{aligned}\min_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} i_{\pm}[\,\phi(X_1,\, X_2)\,] &= r[\,A,\, B_1\,] - r(M) + r(N) \\ &\quad + \max\{\, i_{\pm}(M_1) - r(N_1),\quad i_{\pm}(M_2) - r(N_2)\,\},\end{aligned} \tag{3.10}$$

where

$$\begin{aligned} s_1 &= r(M_1) - 2r(N_1), \quad s_2 = r(M_2) - 2r(N_2), \\ s_3 &= i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2), \\ s_4 &= i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2). \end{aligned}$$

Proof Under (3.6), applying Lemma 1.12 to the variable matrix X_1 in (3.5) and simplifying, we obtain

$$\begin{aligned}\max_{X_1} r[\,\phi(X_1,\, X_2)\,] &= \min\left\{ r[\,A - B_2X_2C_2 - (B_2X_2C_2)^*,\, B_1\,],\ \ r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix}\right\} \\ &= \min\left\{ r[\,A,\, B_1\,],\ \ r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix}\right\},\end{aligned} \tag{3.11}$$

$$\min_{X_1} r[\,\phi(X_1,\, X_2)\,] = 2r[\,A - B_2X_2C_2 - (B_2X_2C_2)^*,\, B_1\,] + r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix}$$

$$\max_{X_1} i_{\pm}[\,\phi(X_1,\, X_2)\,] = i_{\pm}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix}, \tag{3.12}$$

$$\begin{aligned}&- 2r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & B_1 \\ C_1 & 0 \end{bmatrix} \\ &= 2r[\,A,\, B_1\,] + r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\min_{X_1} i_{\pm}[\,\phi(X_1,\, X_2)\,] &= r[\,A - B_2X_2C_2 - (B_2X_2C_2)^*,\, B_1\,] + i_{\pm}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix} \\ &\quad - r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & B_1 \\ C_1 & 0 \end{bmatrix} \\ &= r[\,A,\, B_1\,] + i_{\pm}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix}.\end{aligned} \tag{3.14}$$

Notice that

$$\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} A & C_1^* \\ C_1 & 0 \end{bmatrix} - \begin{bmatrix} B_2 \\ 0 \end{bmatrix} X_2[\, C_2, \, 0\,] - \begin{bmatrix} C_2^* \\ 0 \end{bmatrix} X_2^*[\, B_2^*, \, 0\,]$$
$$:= \psi(X_2). \tag{3.15}$$

Applying Lemma 1.11 to this expression gives

$$\max_{X_2 \in \mathbb{C}^{m\times p_2}} r[\psi(X_2)] = \min\{\, r(N), \quad r(M_1), \quad r(M_2)\,\}, \tag{3.16}$$
$$\min_{X_2 \in \mathbb{C}^{m\times p_2}} r[\psi(X_2)] = 2r(N) + \max\{\, s_1, \quad s_2, \quad s_3, \quad s_4\,\}, \tag{3.17}$$
$$\max_{X_2 \in \mathbb{C}^{m\times p_2}} i_{\pm}[\psi(X_2)] = \min\{\, i_{\pm}(M_1), \quad i_{\pm}(M_2)\,\}, \tag{3.18}$$
$$\min_{X_2 \in \mathbb{C}^{m\times p_2}} i_{\pm}[\psi(X_2)] = r(N) + \max\{\, i_{\pm}(M_1) - r(N_1), \quad i_{\pm}(M_2) - r(N_2)\,\}, \tag{3.19}$$

where

$$s_1 = r(M_1) - 2r(N_1), \quad s_2 = r(M_2) - 2r(N_2),$$
$$s_3 = i_+(M_1) + i_-(M_2) - r(N_1) - r(N_2), \quad s_4 = i_-(M_1) + i_+(M_2) - r(N_1) - r(N_2).$$

Substituting these results into (3.11)–(3.14) yields (3.7)–(3.10). $\square$

It is obviously of great importance to be able to give analytical formulas for calculating the global maximal and minimal ranks and inertias of the matrix expression in (3.5) under the assumptions in (3.7). However, it is not easy to find the global maximal and minimal ranks and inertias of a general $\phi(X_1, \, X_2)$ as given in (3.5). For convenience of representation, we rewrite (3.4) as

$$A_1 - B_1XB_1^* = A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*, \tag{3.20}$$

where

$$A = A_1 - B_1X_0B_1^*, \quad G_1 = B_1, \quad G_2 = F_BB_1^*, \quad G_3 = B_1F_{B_2}, \quad G_4 = F_{B_3}B_1^*. \tag{3.21}$$

It is easy to verify that the above matrices satisfy the conditions

$$(G_2^*) \subseteq \mathscr{R}(G_1), \quad \mathscr{R}(G_3) \subseteq \mathscr{R}(G_1), \quad \mathscr{R}(G_4^*) \subseteq \mathscr{R}(G_1), \quad \mathscr{R}(G_2^*) \subseteq \mathscr{R}(G_3), \quad \mathscr{R}(G_2^*) \subseteq \mathscr{R}(G_4^*). \tag{3.22}$$

In this case, applying Lemma 3.2 to (3.22) yields the main results of this section.

Theorem 3.3 *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1, \, 2, \, 3,$ and assume that the pair of matrix equations*

$$B_2XB_2^* = A_2 \quad \textit{and} \quad B_3XB_3^* = A_3 \tag{3.23}$$

have a common solution $X \in \mathbb{C}_{\mathrm{H}}^{n}$. Also denote the set of all their common Hermitian solutions by

$$\mathcal{S} = \{X \in \mathbb{C}_{\mathrm{H}}^{n} \mid B_2XB_2^* = A_2, \quad B_3XB_3^* = A_3\,\}. \tag{3.24}$$

and let

$$P_1 = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ B_1^* & 0 & B_2^* & B_3^* \end{bmatrix}, \quad P_2 = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ B_1^* & B_2^* & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_3 & B_3 \\ B_1^* & B_3^* & 0 \end{bmatrix}, \tag{3.25}$$

$$Q_1 = \begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ B_1^* & B_2^* & B_3^* & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} A_1 & 0 & B_1 & B_1 \\ 0 & -A_2 & B_2 & 0 \\ B_1^* & B_2^* & 0 & 0 \\ 0 & 0 & 0 & B_3 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} A_1 & 0 & B_1 & B_1 \\ 0 & -A_3 & B_3 & 0 \\ B_1^* & B_3^* & 0 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix}. \tag{3.26}$$

Then,

(a) *The global maximum rank of $A_1 - B_1XB_1^*$ subject to (3.24) is*

$$\max_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^*\,)$$
$$= \min\left\{ r[\, A_1, \, B_1\,], \; r(Q_1) - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(B_2) - r(B_3), \; r(P_2) - 2r(B_2), \; r(P_3) - 2r(B_3) \right\}. \tag{3.27}$$

(b) *The global minimum rank of* $A_1 - B_1XB_1^*$ *subject to* (3.24) *is*

$$\min_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) \\ = 2r[\,A_1,\, B_1\,] - 2r(P_1) + 2r(Q_1) + \max\{\,r(P_2) - 2r(Q_2),\;\; r(P_3) - 2r(Q_3),\;\; u_1,\;\; u_2\,\}, \tag{3.28}$$

where

$$u_1 = i_+(P_2) + i_-(P_3) - r(Q_2) - r(Q_3), \quad u_2 = i_-(P_2) + i_+(P_3) - r(Q_2) - r(Q_3).$$

(c) *The global maximum partial inertia of* $A_1 - B_1XB_1^*$ *subject to* (3.24) *is*

$$\max_{X\in\mathcal{S}} i_\pm(\,A_1 - B_1XB_1^*\,) = \min\{i_\pm(P_2) - r(B_2), \quad i_\pm(P_3) - r(B_3)\}\,. \tag{3.29}$$

(d) *The global minimum partial inertia of* $A_1 - B_1XB_1^*$ *subject to* (3.24) *is*

$$\min_{X\in\mathcal{S}} i_\pm(\,A_1 - B_1XB_1^*\,) = r[\,A_1,\, B_1\,] - r(P_1) + r(Q_1) \\ + \max\{i_\pm(P_2) - r(Q_2),\;\; i_\pm(P_3) - r(Q_3)\}. \tag{3.30}$$

Proof Under (3.22), we find by Lemma 3.2 that

$$\max_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) = \max_{V,\,U} r\,[A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*] \\ = \min\left\{r[\,A,\, G_1\,],\;\; r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix},\;\; r\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix},\;\; r\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix}\right\}, \tag{3.31}$$

$$\min_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) = \min_{V,\,U} r\,[\,A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*\,] \\ = 2r[\,A,\, G_1\,] - 2r\begin{bmatrix} A & G_1 \\ G_2 & 0 \end{bmatrix} + 2r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix} \\ + \max\{s_1,\;\; s_2,\;\; s_3,\;\; s_4\,\}, \tag{3.32}$$

$$\max_{X\in\mathcal{S}} i_\pm(\,A_1 - B_1XB_1^*\,) = \max_{V,\,U} i_\pm\,[A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*] \\ = \min\left\{i_\pm\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix},\;\; i_\pm\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix}\right\}, \tag{3.33}$$

$$\min_{X\in\mathcal{S}} i_\pm(\,A_1 - B_1XB_1^*\,) = \min_{V,\,U} i_\pm\,[\,A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*\,] \\ = r[\,A,\, G_1\,] - r\begin{bmatrix} A & G_1 \\ G_2 & 0 \end{bmatrix} + r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix} + \max\{t_1,\;\; t_2\,\}, \tag{3.34}$$

where

$$s_1 = r\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} - 2r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix},$$

$$s_3 = i_+\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} + i_-\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix},$$

$$s_4 = i_-\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} + i_+\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix},$$

$$t_1 = i_\pm\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix},$$

$$t_2 = i_\pm\begin{bmatrix} A & G_4 \\ G_4^* & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_4^* & 0 & 0 \end{bmatrix}.$$

Applying (1.16)–(1.18) and (1.25), and simplifying by $[\, B_2X_0B_2^*,\, B_3X_0B_3^* \,] = [\, A_2,\, A_3 \,]$, elementary matrix operations and congruence matrix operations, we obtain

$$r[\, A,\, G_1 \,] = r[\, A_1 - B_1X_0B_1^*,\, B_1 \,] = r[\, A_1,\, B_1 \,], \tag{3.35}$$

$$\begin{aligned}
r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1F_{B_2} & B_1F_{B_3} \\ F_BB_1^* & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 & B_1 & 0 \\ B_1^* & 0 & 0 & B^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - r(B) - r(B_2) - r(B_3) \\
&= r\begin{bmatrix} A_1 & B_1 & B_1 & B_1X_0B^* \\ B_1^* & 0 & 0 & B^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - r(B) - r(B_2) - r(B_3) \\
&= r\begin{bmatrix} A_1 & B_1 & B_1 & 0 & 0 \\ B_1^* & 0 & 0 & B_2^* & B_3^* \\ 0 & B_2 & 0 & -A_2 & 0 \\ 0 & 0 & B_3 & 0 & -A_3 \end{bmatrix} - r(B) - r(B_2) - r(B_3) \\
&= r(Q_1) - r(B) - r(B_2) - r(B_3),
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
r\begin{bmatrix} A & G_1 \\ G_2 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 \\ F_BB_1^* & 0 \end{bmatrix} = r\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B^* \end{bmatrix} - r(B) \\
&= r(P_1) - r(B),
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
i_{\pm}\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} &= i_{\pm}\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1F_{B_2} \\ F_{B_2}B_1^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} - r(B_2) \\
&= i_{\pm}\begin{bmatrix} A_1 & B_1 & B_1X_0B_2^*/2 \\ B_1^* & 0 & B_2^* \\ B_1X_0B_2^*/2 & B_2 & 0 \end{bmatrix} - r(B_2) = i_{\pm}\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & -A_2 \end{bmatrix} - r(B_2) \\
&= i_{\pm}(P_2) - r(B_2),
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1F_{B_2} & B_1F_{B_3} \\ F_{B_2}B_1^* & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 & B_1 & 0 \\ B_1^* & 0 & 0 & B_2^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - 2r(B_2) - r(B_3) \\
&= r\begin{bmatrix} A_1 & B_1 & B_1 & B_1X_0B_2^* \\ B_1^* & 0 & 0 & B_2^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - 2r(B_2) - r(B_3) \\
&= r\begin{bmatrix} A_1 & B_1 & B_1 & 0 \\ B_1^* & 0 & 0 & B_2^* \\ 0 & B_2 & 0 & -A_2 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - 2r(B_2) - r(B_3) \\
&= r(Q_2) - 2r(B_2) - r(B_3).
\end{aligned} \tag{3.39}$$

By a similar approach, we can obtain

$$i_{\pm}\begin{bmatrix} A & G_4 \\ G_4^* & 0 \end{bmatrix} = i_{\pm}(P_3) - r(B_3), \quad r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix} = r(Q_3) - r(B_2) - 2r(B_3). \tag{3.40}$$

Substituting (3.35)–(3.40) into (3.31)–(3.34) yields (3.27)–(3.30). $\square$

Some direct consequences of the previous theorem are given below.

Corollary 3.4 *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1,\, 2,\, 3,$ and suppose that each pair of* $B_1XB_1^* =$

A_1, $B_2XB_2^* = A_2$ *and* $B_3XB_3^* = A_3$ *have a common Hermitian solution. Also let* $\mathcal{S}$ *be of the form* (3.23)*. Then,*

$$\max_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^*\,) = \min\left\{ r(B_1),\ r(Q_1) - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(B_2) - r(B_3),\right.$$
$$\left. 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - 2r(B_2),\ 2r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} - 2r(B_3) \right\}, \tag{3.41}$$

$$\min_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^*\,) = 2r(Q_1) - 2r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} - 2r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix}, \tag{3.42}$$

$$\max_{X\in\mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^*\,) = \min\left\{ r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(B_2),\ \ r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} - r(B_3) \right\}, \tag{3.43}$$

$$\min_{X\in\mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^*\,) = r(Q_1) - r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} - r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix}, \tag{3.44}$$

where Q_1 *is of the form* (3.26)*.*

Proof Under the given conditions, the ranks and inertias of the block matrices in (3.25) and (3.26) are given by

$$r(P_1) = r(B_1) + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},\quad r(P_2) = 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ r(P_3) = 2r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix},\ i_{\pm}(P_2) = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ i_{\pm}(P_3) = r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix},$$

$$r(Q_2) = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\quad r(Q_3) = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix}.$$

Hence (3.27)–(3.30) reduce to (3.41)–(3.44). □

Corollary 3.5 *Let* $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i\times m_i}$ *and* $B_i \in \mathbb{C}^{m_i\times n}$ *be given for* $i = 1,\ 2,\ 3,$ *and suppose that each pair of the triple matrix equations*

$$B_1XB_1^* = A_1,\quad B_2XB_2^* = A_2,\quad B_3XB_3^* = A_3 \tag{3.45}$$

have a common Hermitian solution. Then, there exists a Hermitian X *such that* (3.45) *holds if and only if*

$$r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ B_1^* & B_2^* & B_3^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r[\, B_1^*,\ B_2^*,\ B_3^*\,]. \tag{3.46}$$

Proof It follows from (3.42). □

A challenging open problem on the triple matrix equations in (3.45) is to give a parametric form for their general common Hermitian solution.

Setting $B_1 = I_n$ in Theorem 3.3 may yield a group of results on the extremal ranks and inertias of $A_1 - X$ subject to (3.24). In particular, we have the following consequences.

Corollary 3.6 *Let* $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ *and* $B_i \in \mathbb{C}^{m_i\times n}$ *be given for* $i = 2,\ 3,$ *and assume that* (3.23) *has a common solution. Also let* $\mathcal{S}$ *be of the form* (3.24)*. Then,*

(a) *The global maximum rank of the solution of* (3.24) *is*

$$\max_{X\in\mathcal{S}} r(X) = \min\{\, n,\quad s_1,\quad s_2,\quad s_3\,\}, \tag{3.47}$$

where

$$s_1 = 2n + r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(B_2) - r(B_3),$$

$$s_2 = 2n + r(A_2) - 2r(B_2),\quad s_3 = 2n + r(A_3) - 2r(B_3).$$

(b) *The global minimum rank of the solution of* (3.24) *is*

$$\min_{X\in\mathcal{S}} r(X) = 2r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} + \max\{\, t_1, \quad t_2, \quad t_3, \quad t_4 \,\}, \tag{3.48}$$

where

$$t_1 = r(A_2) - 2r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}, \quad t_2 = r(A_3) - 2r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix},$$
$$t_3 = i_{+}(A_2) + i_{-}(A_3) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix},$$
$$t_4 = i_{-}(A_2) + i_{+}(A_3) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix}.$$

(c) *The global maximum partial inertia of the solution of* (3.24) *is*

$$\max_{X\in\mathcal{S}} i_{\pm}(X) = \min\{\, n + i_{\pm}(A_2) - r(B_2), \quad n + i_{\pm}(A_3) - r(B_3) \,\}. \tag{3.49}$$

(d) *The global minimum partial inertia of the solution of* (3.24) *is*

$$\min_{X\in\mathcal{S}} i_{\pm}(X) = r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} + \max\left\{ i_{\pm}(A_2) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}, \; i_{\pm}(A_3) - r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix} \right\}. \tag{3.50}$$

In consequence,

(e) *Eq.* (3.23) *has a solution* $X > 0$ *if and only if*

$$A_2 \geqslant 0, \quad A_3 \geqslant 0, \;\; \mathscr{R}(A_2) = \mathscr{R}(B_2), \quad \mathscr{R}(A_3) = \mathscr{R}(B_3).$$

(f) *All solutions of* (3.23) *satisfy* $X > 0$ *if and only if* $A_2 \geqslant 0$, $A_3 \geqslant 0$ *and one of*

$$r(A_2) = r(B_2) = n, \quad r(A_3) = r(B_3) = n.$$

(g) *Eq.* (3.23) *has a solution* $X < 0$ *if and only if*

$$A_2 \leqslant 0, \quad A_3 \leqslant 0, \quad \mathscr{R}(A_2) = \mathscr{R}(B_2), \quad \mathscr{R}(A_3) = \mathscr{R}(B_3).$$

(h) *All solutions of* (3.23) *satisfy* $X < 0$ *if and only if* $A_2 \leqslant 0$, $A_3 \leqslant 0$ *and one of*

$$r(A_2) = r(B_2) = n, \quad r(A_3) = r(B_3) = n.$$

(i) *Eq.* (3.23) *has a solution* $X \geqslant 0$ *if and only if*

$$A_2 \geqslant 0, \quad A_3 \geqslant 0, \quad \mathscr{R}\begin{bmatrix} A_2 \\ 0 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix}, \quad \mathscr{R}\begin{bmatrix} 0 \\ A_3 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}.$$

(j) *All solutions of* (3.23) *satisfy* $X \geqslant 0$ *if and only if* $A_2 \geqslant 0$, $A_3 \geqslant 0$ *and one of*

$$r(B_2) = n \;\; \text{and} \;\; r(B_3) = n.$$

(k) *Eq.* (3.23) *has a solution* $X \leqslant 0$ *if and only if*

$$A_2 \leqslant 0, \quad A_3 \leqslant 0, \quad \mathscr{R}\begin{bmatrix} A_2 \\ 0 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix}, \quad \mathscr{R}\begin{bmatrix} 0 \\ A_3 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}.$$

(l) *All solutions of* (3.23) *satisfy* $X \leqslant 0$ *if and only if* $A_2 \leqslant 0$, $A_3 \leqslant 0$ *and one of*

$$r(B_2) = n \;\; \text{and} \;\; r(B_3) = n.$$

Proof Set $A_1 = 0$ and $B_1 = I_n$ in Theorem 3.3 and simplifying, we obtain (a)–(d). Applying Lemma 1.5 to (3.48) and (3.49), we obtain (e)–(l). $\square$

Corollary 3.6(e)–(l) give a set of analytical characterizations for the existence of definite common solutions of the two matrix equations in (3.23) by using some rank and range equalities and inequalities. These characterizations are simple and easy to understand in comparison with various known conditions (see, e.g., [14, 40, 41])s on the existence of definite common solutions of (3.23).

Rewrite $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ as

$$[\,B_{21},\, B_{22}\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_{21}^* \\ B_{22}^* \end{bmatrix} = A_2, \quad [\,B_{31},\, B_{32}\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_{31}^* \\ B_{32}^* \end{bmatrix} = A_3, \tag{3.51}$$

where $B_{i1} \in \mathbb{C}^{m_i\times n_1}$, $B_{i2} \in \mathbb{C}^{m_i\times n_2}$, $i = 2,\, 3$, $X_1 \in \mathbb{C}_{\mathrm{H}}^{n_1}$, $X_2 \in \mathbb{C}^{n_1\times n_2}$ and $X_3 \in \mathbb{C}_{\mathrm{H}}^{n_2}$ with $n_1 + n_2 = n$. We next derive the extremal ranks and inertias of the submatrices X_1 and X_3 in a Hermitian solution of (3.51). Note that $X_1,\, X_2,\, X_3$ in (3.51) can be rewritten as

$$X_1 = P_1XP_1^*, \quad X_2 = P_1XP_2^*, \quad X_3 = P_2XP_2^*, \tag{3.52}$$

where $P_1 = [\, I_{n_1},\, 0\,]$ and $P_2 = [\,0,\, I_{n_2}\,]$. For convenience, we adopt the following notation for the collections of the submatrices X_1 and X_3 in (3.51):

$$\mathcal{S}_1 = \{X_1 = P_1XP_1^* \mid B_2XB_2^* = A_2,\ B_3XB_3^* = A_3,\ X = X^*\}\,, \tag{3.53}$$

$$\mathcal{S}_3 = \{X_3 = P_2XP_2^* \mid B_2XB_2^* = A_2,\ B_3XB_3^* = A_3,\ X = X^*\}\,. \tag{3.54}$$

The global maximal and minimal ranks and partial inertias of the submatrices X_1 and X_3 in (3.51) can easily be derived from Theorem 3.3. The details are omitted.

If each of the triple matrix equations in (1.8) is not consistent, people may alternatively seek its common approximation solutions under various given optimal criteria. One of the most useful approximation solutions of $BXB^* = A$ is the well-known least-squares Hermitian solution, which is defined to be a Hermitian matrix X that minimizes the objective function:

$$\|\, A - BXB^*\,\|^2 = \mathrm{tr}[\,(\,A - BXB^*\,)(\,A - BXB^*\,)^*\,]. \tag{3.55}$$

The normal equation corresponding to the norm minimization problem is given by

$$B^*BXB^*B = B^*AB. \tag{3.56}$$

This equation is always consistent. Concerning the common least-squares Hermitian solution of (1.8), we have the following result.

Corollary 3.7 *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1,\, 2,\, 3$. Then, triple matrix equations have a common least-squares Hermitian solution, namely, there exists an $X \in \mathbb{C}_{\mathrm{H}}^{n\times n}$ such that*

$$\|\, A_i - B_iXB_i^*\,\| = \min, \quad i = 1,\, 2,\, 3, \tag{3.57}$$

if and only if

$$r\begin{bmatrix} B_i^*A_iB_i & 0 & B_i^*B_i \\ 0 & -B_j^*A_jB_j & B_j^*B_j \\ B_i^*B_i & B_j^*B_j & 0 \end{bmatrix} = 2r\begin{bmatrix} B_i \\ B_j \end{bmatrix}, \quad i \neq j,\ i,\, j = 1,\, 2,\, 3, \tag{3.58}$$

$$r\begin{bmatrix} B_1^*A_1B_1 & 0 & 0 & B_1^*B_1 & B_1^*B_1 \\ 0 & -B_2^*A_2B_2 & 0 & B_2^*B_2 & 0 \\ 0 & 0 & -B_3^*A_3B_3 & 0 & B_3^*B_3 \\ B_1^*B_1 & B_2^*B_2 & B_3^*B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}. \tag{3.59}$$

Proof It follows from Lemma 3.1, Corollary 3.5 and (3.56). $\square$

4 The extremal ranks and inertias of $A_1 - B_1XB_1^*$ subject to the Hermitian solutions of $B_4X = A_4$

Also $B_4X = A_4$ in (1.3) is not given in symmetric pattern, it may have a Hermitian solution, as shown in Theorem 1.10. So that the global extremal ranks and inertias of $A_1 - B_1XB_1^*$ subject to the Hermitian solution or nonnegative definite solution of $B_4X = A_4$ can also be derived.

Theorem 4.1 *Assume that the matrix equation $B_4X = A_4$ in (1.3) has a Hermitian solution, i.e., $\mathscr{R}(A_4) \subseteq \mathscr{R}(B_4)$ and $A_4B_4^* = B_4A_4^*$, and let*

$$\mathcal{S} = \{\, X \in \mathbb{C}_{\rm H}^n \mid B_4X = A_4 \,\}, \quad M = \begin{bmatrix} A_1 & B_1 \\ A_4B_1^* & B_4 \end{bmatrix}, \quad N = \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_4^* \\ 0 & B_4 & -A_4B_4^* \end{bmatrix}. \tag{4.1}$$

Then,

$$\max_{X \in \mathcal{S}} r(\, A_1 - B_1XB_1^* \,) = r(M) - r(B_4), \tag{4.2}$$

$$\min_{X \in \mathcal{S}} r(\, A_1 - B_1XB_1^* \,) = 2r(M) - r(N), \tag{4.3}$$

$$\max_{X \in \mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^* \,) = i_{\pm}(N) - r(B_4), \tag{4.4}$$

$$\min_{X \in \mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^* \,) = r(M) - i_{\mp}(N). \tag{4.5}$$

In consequences,

(a) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\rm H}^n$ such that $A_1 - B_1XB_1^*$ is nonsingular if and only if $r(M) = r(B_4) + m_1$.*

(b) *$A_1 - B_1XB_1^*$ is nonsingular for all Hermitian solution of $B_4X = A_4$ if and only if $2r(M) = r(N) + m_1$.*

(c) *The pair of matrix equations $B_1XB_1^* = A_1$ and $B_4X = A_4$ have a common Hermitian solution if and only if $\mathscr{R}\begin{bmatrix} A_1 \\ A_4B_1^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} B_1 \\ B_4 \end{bmatrix}$.*

(d) *$B_1XB_1^* = A_1$ holds for all Hermitian solutions of $B_4X = A_4$ if and only if $r(M) = r(B_4)$.*

(e) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\rm H}^n$ such that $A_1 - B_1XB_1^* > 0$ ($A_1 - B_1XB_1^* < 0$) if and only if $i_{+}(N) = r(B_4) + m_1$ ($i_{-}(N) = r(B_4) + m_1$).*

(f) *$A_1 - B_1XB_1^* > 0$ ($A_1 - B_1XB_1^* < 0$) holds for all Hermitian solutions of $B_4X = A_4$ if and only if $r(M) = i_{-}(N) + m_1$ ($r(M) = i_{+}(N) + m_1$).*

(g) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\rm H}^n$ such that $A_1 - B_1XB_1^* \geqslant 0$ ($A_1 - B_1XB_1^* \leqslant 0$) if and only if $r(M) = i_{+}(N)$ $(r(M) = i_{-}(N))$.*

(f) *$A_1 - B_1XB_1^* \geqslant 0$ ($A_1 - B_1XB_1^* \geqslant 0$) holds for all Hermitian solutions of $B_4X = A_4$ if and only if $i_{-}(N) = r(B_4)$ ($i_{+}(N) = r(B_4)$).*

Proof. From Lemma 1.10(a), the general Hermitian solution of $B_4X = A_4$ can be written as

$$X = B_4^{\dagger}A_4 + (B_4^{\dagger}A_4)^* - B_4^{\dagger}A_4B_4^{\dagger}B_4 + F_{B_4}WF_{B_4}, \tag{4.6}$$

where $W \in \mathbb{C}_{\rm H}^n$ is arbitrary. Substituting (4.5) into $A_1 - B_1XB_1^*$ gives

$$A_1 - B_1XB_1^* = P - B_1F_{B_4}WF_{B_4}B_1^*, \tag{4.7}$$

where $G = A_1 - B_1B_4^{\dagger}A_4B_1^* - B_1(B_4^{\dagger}A_4)^*B_1^* + B_1B_4^{\dagger}A_4B_4^{\dagger}B_4B_1^*$. Applying (1.33)–(1.36) to (4.6) yields

$$\max_{X \in \mathcal{S}} r(\, A_1 - B_1XB_1^* \,) = \max_{W \in \mathbb{C}_{\rm H}^n} r(\, G - B_1F_{B_4}WF_{B_4}B_1^* \,) = r[\, G,\ B_1F_{B_4} \,], \tag{4.8}$$

$$\min_{X \in \mathcal{S}} r(\, A_1 - B_1XB_1^* \,) = \min_{W \in \mathbb{C}_{\rm H}^n} r(\, G - B_1F_{B_4}WF_{B_4}B_1^* \,) = 2r[\, G,\ B_1F_{B_4} \,] - r\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix}, \tag{4.9}$$

$$\max_{X \in \mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^* \,) = \max_{W \in \mathbb{C}_{\rm H}^n} r(\, G - B_1F_{B_4}WF_{B_4}B_1^* \,) = i_{\pm}\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix}, \tag{4.10}$$

$$\min_{X \in \mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^* \,) = \min_{W \in \mathbb{C}_{\rm H}^n} r(\, G - B_1F_{B_4}WF_{B_4}B_1^* \,) = r[\, G,\ B_1F_{B_4} \,] - i_{\mp}\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix}. \tag{4.11}$$

It is easy to verify that under $B_4B_4^{\dagger}A_4 = A_4$, the equality $B_4(B_4^{\dagger}A_4)^* = B_4A_4^*(B_4^{\dagger})^* = A_4B_4^*(B_4^{\dagger})^* = A_4B_4^{\dagger}B_4$ holds. In this case, applying It is easy to verify by (1.17) and (1.25) to (4.8)–(4.11) and simplifying by elementary matrix operations and congruence matrix operations, we obtain

$$\begin{aligned} r[\, G,\ B_1F_{B_4} \,] &= r\begin{bmatrix} A_1 - B_1B_4^{\dagger}A_4B_1^* - B_1(B_4^{\dagger}A_4)^*B_1^* + B_1B_4^{\dagger}A_4B_4^{\dagger}B_4B_1^* & B_1 \\ 0 & B_4 \end{bmatrix} - r(B_4) \\ &= r\begin{bmatrix} A_1 & B_1 \\ A_4B_1^* + B_4(B_4^{\dagger}A_4)^*B_1^* - A_4B_4^{\dagger}B_4B_1^* & B_4 \end{bmatrix} - r(B_4) \\ &= r\begin{bmatrix} A_1 & B_1 \\ A_4B_1^* & B_4 \end{bmatrix} - r(B_4) = r(M) - r(B_4), \end{aligned} \tag{4.12}$$

$$
\begin{aligned}
& i_\pm\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix} \\
&= i_\pm\begin{bmatrix} A_1 - B_1B_4^\dagger A_4B_1^* - B_1(B_4^\dagger A_4)^*B_1^* + B_1B_4^\dagger A_4B_4^\dagger B_4B_1^* & B_1 & 0 \\ B_1^* & 0 & B_4^* \\ 0 & B_4 & 0 \end{bmatrix} - r(B_4) \\
&= i_\pm\begin{bmatrix} A_1 & B_1 & \frac{1}{2}B_1B_4^\dagger A_4B_4^* + \frac{1}{2}B_1A_4^* - \frac{1}{2}B_1B_4^\dagger A_4B_4^* \\ B_1^* & 0 & B_4^* \\ \frac{1}{2}A_4B_1^* + \frac{1}{2}B_4B_1(B_4^\dagger A_4)^*B_1^* - \frac{1}{2}A_4B_4^\dagger B_4B_1^* & B_4 & 0 \end{bmatrix} \\
&\quad - r(B_4) \\
&= i_\pm\begin{bmatrix} A_1 & B_1 & \frac{1}{2}B_1A_4^* \\ B_1^* & 0 & B_4^* \\ \frac{1}{2}A_4B_1^* & B_4 & 0 \end{bmatrix} - r(B_4) = i_\pm\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_4^* \\ 0 & B_4 & -A_4B_4^* \end{bmatrix} - r(B_4) = i_\pm(N) - r(B_4). \qquad (4.13)
\end{aligned}
$$

Substituting (4.12) and (4.13) into (4.8)–(4.11) yields (4.2)–(4.5). Applying Lemma 1.5 to (4.2)–(4.5) yields (a)–(f). $\square$

Theorem 4.2 *Assume that the matrix equation $B_4X = A_4$ in* (1.3) *has a nonnegative definite solution, i.e., $\mathscr{R}(A_4) \subseteq \mathscr{R}(B_4)$, $A_4B_4^* \geqslant 0$ and $r(A_4B_4^*) = r(A_4)$, and let*

$$
\mathcal{S} = \{\, 0 \leqslant X \in \mathbb{C}_{\mathrm{H}}^n \mid A_4X = B_4 \,\},\ M_1 = \begin{bmatrix} A_1 & B_1 \\ A_4B_1^* & B_4 \end{bmatrix},\ M_2 = \begin{bmatrix} A_1 & B_1A_4^* \\ A_4B_1^* & A_4B_4^* \end{bmatrix},\ N = \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_4^* \\ 0 & B_4 & -A_4B_4^* \end{bmatrix}. \qquad (4.14)
$$

Then,

$$
\max_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^* \,) = r(M_1) - r(B_4), \qquad (4.15)
$$

$$
\min_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^* \,) = r(M_1) + i_-(M_2) - i_-(N), \qquad (4.16)
$$

$$
\max_{X\in\mathcal{S}} i_+(\, A_1 - B_1XB_1^* \,) = i_+(M_2) - r(A_4), \qquad (4.17)
$$

$$
\min_{X\in\mathcal{S}} i_+(\, A_1 - B_1XB_1^* \,) = r(M_1) - i_-(N), \qquad (4.18)
$$

$$
\max_{X\in\mathcal{S}} i_-(\, A_1 - B_1XB_1^* \,) = i_-(N) - r(A_4), \qquad (4.19)
$$

$$
\min_{X\in\mathcal{S}} i_-(\, A_1 - B_1XB_1^* \,) = i_-(M_2). \qquad (4.20)
$$

In consequences,

(a) *$B_4X = A_4$ has a nonnegative definite solution such that $A_1 - B_1XB_1^*$ is nonsingular if and only if $r(M_1) = r(B_4) + m_1$.*

(b) *$A_1 - B_1XB_1^*$ is nonsingular for all nonnegative definite solution of $B_4X = A_4$ if and only if $r(M_1) + i_-(M_2) = i_-(N) + m_1$.*

(c) *The pair of matrix equations $B_1XB_1^* = A_1$ and $B_4X = A_4$ have a common nonnegative definite solution if and only if $r(M_1) + i_-(M_2) = i_-(N)$.*

(d) *$B_1XB_1^* = A_1$ holds for all nonnegative definite solutions of $B_4X = A_4$ if and only if $r(M) = r(B_4)$.*

(e) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\mathrm{H}}^n$ such that $A_1 - B_1XB_1^* > 0$ if and only if $i_+(M_2) = r(A_4) + m_1$.*

(f) *$A_1 - B_1XB_1^* > 0$ holds for all Hermitian solutions of $B_4X = A_4$ if and only if $r(M_1) = i_-(N) + m_1$.*

(g) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\mathrm{H}}^n$ such that $A_1 - B_1XB_1^* < 0$ if and only if $i_-(N) = r(A_4) + m_1$.*

(h) *$A_1 - B_1XB_1^* < 0$ holds for all Hermitian solutions of $B_4X = A_4$ if and only if $i_-(M_2) = m_1$.*

(i) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\mathrm{H}}^n$ such that $A_1 - B_1XB_1^* \geqslant 0$ if and only if $M_2 \geqslant 0$.*

(j) *$A_1 - B_1XB_1^* \geqslant 0$ holds for all Hermitian solutions of $B_4X = A_4$ if and only if $i_-(N) = r(A_4)$.*

(k) *$B_4X = A_4$ has a solution $X \in \mathbb{C}_{\mathrm{H}}^n$ such that $A_1 - B_1XB_1^* \leqslant 0$ if and only if $r(M_1) = i_-(N)$.*

(l) *$A_1 - B_1XB_1^* \leqslant 0$ holds for all Hermitian solutions of $B_4X = A_4$ if and only if $i_+(M_2) = r(A_4)$.*

Proof. From Lemma 1.10(b), the general nonnegative definite solution of $B_4X = A_4$ can be written as

$$X = A_4^*(A_4B_4^*)^{\dagger}A_4 + F_{B_4}WF_{B_4}, \tag{4.21}$$

where $0 \leqslant W \in \mathbb{C}_{\mathrm{H}}^n$ is arbitrary. Substituting (4.21) into $A_1 - B_1XB_1^*$ gives

$$A_1 - B_1XB_1^* = G - B_1F_{B_4}WF_{B_4}B_1^*, \tag{4.22}$$

where $G = A_1 - B_1A_4^*(A_4B_4^*)^{\dagger}A_4B_1^*$. Applying (1.40)–(1.42) to (4.22) yields

$$\max_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) = \max_{0\leqslant W\in\mathbb{C}_{\mathrm{H}}^n} r(\,G - B_1F_{B_4}WF_{B_4}B_1^*\,) = r[\,G,\, B_1F_{B_4}\,], \tag{4.23}$$

$$\min_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) = \min_{0\leqslant W\in\mathbb{C}_{\mathrm{H}}^n} r(\,G - B_1F_{B_4}WF_{B_4}B_1^*\,) = i_-(G) + r[\,G,\, B_1F_{B_4}\,] - i_-\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix}, \tag{4.24}$$

$$\max_{X\in\mathcal{S}} i_+(\,A_1 - B_1XB_1^*\,) = \max_{0\leqslant W\in\mathbb{C}_{\mathrm{H}}^n} r(\,G - B_1F_{B_4}WF_{B_4}B_1^*\,) = i_+(G), \tag{4.25}$$

$$\min_{X\in\mathcal{S}} i_+(\,A_1 - B_1XB_1^*\,) = \max_{0\leqslant W\in\mathbb{C}_{\mathrm{H}}^n} i_+(\,G - B_1F_{B_4}WF_{B_4}B_1^*\,) = r[\,G,\, B_1F_{B_4}\,] - i_-\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix}, \tag{4.26}$$

$$\max_{X\in\mathcal{S}} i_-(\,A_1 - B_1XB_1^*\,) = \max_{0\leqslant W\in\mathbb{C}_{\mathrm{H}}^n} i_-(\,G - B_1F_{B_4}WF_{B_4}B_1^*\,) = i_-\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix}, \tag{4.27}$$

$$\min_{X\in\mathcal{S}} i_{\pm}(\,A_1 - B_1XB_1^*\,) = \min_{0\leqslant W\in\mathbb{C}_{\mathrm{H}}^n} i_-(\,G - B_1F_{B_4}WF_{B_4}B_1^*\,) = i_-(G). \tag{4.28}$$

It is easy to verify by (1.17), (1.23) and (1.25) that

$$r[\,G,\, B_1F_{B_4}\,] = r\begin{bmatrix} A_1 - B_1A_4^*(A_4B_4^*)^{\dagger}A_4B_1^* & B_1 \\ 0 & B_4 \end{bmatrix} - r(B_4) = r\begin{bmatrix} A_1 & B_1 \\ A_4B_1^* & B_4 \end{bmatrix} - r(B_4), \tag{4.29}$$

$$\begin{aligned} i_{\pm}\begin{bmatrix} G & B_1F_{B_4} \\ F_{B_4}B_1^* & 0 \end{bmatrix} &= i_{\pm}\begin{bmatrix} A_1 - B_1A_4^*(A_4B_4^*)^{\dagger}A_4B_1^* & B_1 & 0 \\ B_1^* & 0 & B_4^* \\ 0 & B_4 & 0 \end{bmatrix} - r(B_4) \\ &= i_{\pm}\begin{bmatrix} A_1 & B_1 & \frac{1}{2}B_1A_4^* \\ B_1^* & 0 & B_4^* \\ \frac{1}{2}A_4B_1^* & B_4 & 0 \end{bmatrix} - r(B_4) = i_{\pm}\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_4^* \\ 0 & B_4 & -A_4B_4^* \end{bmatrix} - r(B_4). \end{aligned} \tag{4.30}$$

$$i_{\pm}(G) = i_{\pm}[\,A_1 - B_1A_4^*(A_4B_4^*)^{\dagger}A_4B_1^*\,] = i_{\pm}\begin{bmatrix} A_1 & B_1A_4^* \\ A_4B_1^* & A_4B_4^* \end{bmatrix} - i_{\pm}(A_4B_4^*). \tag{4.31}$$

Substituting (4.29)–(4.31) into (4.23)–(4.28) yields (4.15)–(4.20). Applying Lemma 1.5 to (4.2)–(4.5) yields (a)–(l). $\square$

Corollary 4.3 *Assume that the matrix equation in Lemma 1.10 has a Hermitian solution,* $P \in \mathbb{C}_{\mathrm{H}}^n$, *and let* $\mathcal{S} = \{\, X \in \mathbb{C}_{\mathrm{H}}^n \mid AX = B\,\}$. *Then,*

$$\max_{X\in\mathcal{S}} r(\,X - P\,) = r(\,B - AP\,) - r(A) + n, \tag{4.32}$$

$$\min_{X\in\mathcal{S}} r(\,X - P\,) = 2r(\,B - AP\,) - r(\,BA^* - APA^*\,), \tag{4.33}$$

$$\max_{X\in\mathcal{S}} i_{\pm}(\,X - P\,) = i_{\pm}(\,BA^* - APA^*\,) - r(A) + n, \tag{4.34}$$

$$\min_{X\in\mathcal{S}} i_{\pm}(\,X - P\,) = r(\,B - AP\,) - i_{\mp}(\,BA^* - APA^*\,). \tag{4.35}$$

In consequence,

(a) *There exists an* $X \in \mathcal{S}$ *such that* $X - P$ *is nonsingular if and only if*

$$\mathscr{R}(\,AP - B\,) = \mathscr{R}(A).$$

(b) $X - P$ *is nonsingular for all* $X \in \mathcal{S}$ *if and only if*

$$2r(\,B - AP\,) = r(\,BA^* - APA^*\,) + n.$$

(c) *There exists an* $X \in \mathcal{S}$ *such that* $X > P$ $(X < P)$ *holds if and only if*

$$\mathscr{R}(\,BA^* - APA^*\,) = \mathscr{R}(A) \ \ \text{and} \ \ BA^* \geqslant APA^* \quad (\mathscr{R}(\,BA^* - APA^*\,) = \mathscr{R}(A) \ \ \text{and} \ \ BA^* \leqslant APA^*\,).$$

(d) $X > P$ ($X < P$) *holds for all* $X \in \mathcal{S}$ *if and only if*

$$r(\,B - AP\,) = n \ \ \text{and} \ \ BA^* \geqslant APA^* \quad (\,r(\,B - AP\,) = n \ \ \text{and} \ \ AB^* \leqslant APA^*\,).$$

(e) *There exists an* $X \in \mathcal{S}$ *such that* $X \geqslant P$ ($X \leqslant P$) *holds if and only if*

$$\mathscr{R}(\,B - AP\,) = \mathscr{R}(\,BA^* - APA^*\,) \ \text{and} \ BA^* \geqslant APA^* \ \ (\mathscr{R}(\,B - AP\,) = \mathscr{R}(\,BA^* - APA^*\,) \ \ \text{and} \ \ BA^* \leqslant APA^*\,).$$

(f) $X \geqslant P$ ($X \leqslant P$) *holds for all* $X \in \mathcal{S}$ *if and only if*

$$BA^* \geqslant APA^* \ \ \text{and} \ \ r(A) = n \quad (\,BA^* \leqslant APA^* \ \ \text{and} \ \ r(A) = n\,).$$

Corollary 4.4 *Assume that the matrix equation in Lemma 1.10 has a Hermitian solution* $X \geqslant 0$*, and let* $0 \leqslant P \in \mathbb{C}_{\mathrm{H}}^{n}$*. Also, define*

$$\mathcal{S} = \{\,0 \leqslant X \in \mathbb{C}_{\mathrm{H}}^{n} \mid AX = B\,\}, \quad M = \begin{bmatrix} BA^* & B \\ B^* & P \end{bmatrix}. \tag{4.36}$$

Then,

$$\max_{X \in \mathcal{S}} r(\,X - P\,) = r(\,B - AP\,) - r(A) + n, \tag{4.37}$$

$$\min_{X \in \mathcal{S}} r(\,X - P\,) = i_{-}(M) + r(\,B - AP\,) - i_{+}(\,BA^* - APA^*\,), \tag{4.38}$$

$$\max_{X \in \mathcal{S}} i_{+}(\,X - P\,) = i_{+}(\,BA^* - APA^*\,) - r(A) + n, \tag{4.39}$$

$$\min_{X \in \mathcal{S}} i_{+}(\,X - P\,) = i_{-}(M), \tag{4.40}$$

$$\max_{X \in \mathcal{S}} i_{-}(\,X - P\,) = i_{+}(M) - r(B), \tag{4.41}$$

$$\min_{X \in \mathcal{S}} i_{-}(\,X - P\,) = r(\,B - AP\,) - i_{+}(\,BA^* - APA^*\,). \tag{4.42}$$

In consequence,

(a) *There exists an* $X \in \mathcal{S}$ *such that* $X - P$ *is nonsingular if and only if* $\mathscr{R}(\,B - AP\,) = \mathscr{R}(A)$.

(b) $X - P$ *is nonsingular for all* $X \in \mathcal{S}$ *if and only if* $i_{-}(M) + r(\,B - AP\,) = i_{+}(\,BA^* - APA^*\,) + n$.

(c) *There exists an* $X \in \mathcal{S}$ *such that* $X > P$ *holds if and only if* $\mathscr{R}(\,BA^* - APA^*\,) = \mathscr{R}(A)$ *and* $BA^* \geqslant APA^*$.

(d) $X > P$ *holds for all* $X \in \mathcal{S}$ *if and only if* $i_{-}(M) = r(A)$.

(e) *There exists an* $X \in \mathcal{S}$ *such that* $X < P$ *holds if and only if* $i_{-}(M) = r(B) + n$.

(f) $X < P$ *holds for all* $X \in \mathcal{S}$ *if and only if* $r(\,B - AP\,) = n$ *and* $BA^* \leqslant APA^*$.

(g) *There exists an* $X \in \mathcal{S}$ *such that* $X \geqslant P$ *if and only if* $\mathscr{R}(\,B - AP\,) = \mathscr{R}(\,BA^* - APA^*\,)$ *and* $BA^* \geqslant APA^*$.

(h) $X \geqslant P$ *holds for all* $X \in \mathcal{S}$ *if and only if* $i_{-}(M) = r(B)$.

(i) *There exists an* $X \in \mathcal{S}$ *such that* $X \leqslant P$ *if and only if* $M \geqslant 0$.

(j) $X \leqslant P$ *holds for all* $X \in \mathcal{S}$ *if and only if* $i_{+}(\,BA^* - APA^*\,) = n - r(A)$.

Corollary 4.5 *Assume that the matrix equation in Lemma 1.10 has a Hermitian solution. Then,*

$$\max_{AX = B,\, X \in \mathbb{C}_{\mathrm{H}}^{n}} r(X) = n + r(B) - r(A), \tag{4.43}$$

$$\min_{AX = B,\, X \in \mathbb{C}_{\mathrm{H}}^{n}} r(X) = 2r(B) - r(AB^*), \tag{4.44}$$

$$\max_{AX = B,\, X \in \mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(X) = n + i_{\pm}(AB^*) - r(A), \tag{4.45}$$

$$\min_{AX = B,\, X \in \mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(X) = r(B) - i_{\mp}(AB^*). \tag{4.46}$$

Hence,

(a) $AX = B$ *has a nonsingular Hermitian solution if and only if* $r(A) = r(B)$.

(b) $AX = B$ *has a solution* $X > 0$ ($X < 0$) *if and only if* $AB^* \geqslant 0$ *and* $r(AB^*) = r(A)$ ($AB^* \leqslant 0$ *and* $r(AB^*) = r(A)$).

(c) $AX = B$ *has a solution* $X \geqslant 0$ ($X \leqslant 0$) *if and only if* $AB^* \geqslant 0$ *and* $r(AB^*) = r(B)$ ($AB^* \leqslant 0$ *and* $r(AB^*) = r(B)$).

(d) *The rank of the Hermitian solution of* $AX = B$ *is invariant* $\Leftrightarrow$ *the positive index of inertia of the Hermitian solution of* $AX = B$ *is invariant* $\Leftrightarrow$ *the negative index of inertia of the Hermitian solution of* $AX = B$ *is invariant* $\Leftrightarrow$ $r(AB^*) = r(A) + r(B) - n$.

Finally, we rewrite the matrix equation $AX = B$ as

$$[\,A_1,\ A_2\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} = [\,B_1,\ B_2\,], \tag{4.47}$$

where $A_i \in \mathbb{C}^{m\times n_i}$, $B_i \in \mathbb{C}^{m\times n_i}$, $X_1 \in \mathbb{C}_{\rm H}^{n_1}$, $X_2 \in \mathbb{C}^{n_1\times n_2}$, $X_3 \in \mathbb{C}_{\rm H}^{n_2}$ for $i = 1,\,2$ and $n_1 + n_2 = n$. Note that the unknown submatrices in (4.47) can be written as

$$X_1 = P_1XP_1^*, \quad X_2 = P_1XP_2^*, \quad X_3 = P_2XP_2^*, \tag{4.48}$$

where $P_1 = [\,I_{n_1},\,0\,]$ and $P_2 = [\,0,\,I_{n_2}\,]$. We next find the extremal ranks and inertias of the submatrices X_1 and X_3 in a Hermitian solution of (4.47). For convenience, let

$$\mathcal{S}_1 = \{X_1 \in \mathbb{C}_{\rm H}^{n_1} \mid X_1 = P_1XP_1^*,\ AX = B,\ X \in \mathbb{C}_{\rm H}^{n}\}, \tag{4.49}$$
$$\mathcal{S}_3 = \{X_3 \in \mathbb{C}_{\rm H}^{n_2} \mid X_3 = P_2XP_2^*,\ AX = B,\ X \in \mathbb{C}_{\rm H}^{n}\}. \tag{4.50}$$

Applying Theorem 4.1 to (4.49) and (4.50) gives the following results. The details of the proof are omitted.

Theorem 4.6 *Assume that matrix equation in* (4.17) *has a Hermitian solution, and let* $\mathcal{S}_1$ *and* $\mathcal{S}_3$ *be of the forms in* (4.49) *and* (4.50). *Then, the global maximal and minimal ranks and inertias of the Hermitian matrices in* $\mathcal{S}_1$ *and* $\mathcal{S}_3$ *are given by*

$$\max_{X_1\in\mathcal{S}_1} r(\,X_1\,) = n_1 + r[\,A_2,\,B_1\,] - r(A), \tag{4.51}$$

$$\min_{X_1\in\mathcal{S}_1} r(\,X_1\,) = 2r[\,A_2,\,B_1\,] - r\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix}, \tag{4.52}$$

$$\max_{X_1\in\mathcal{S}_1} i_{\pm}(\,X_1\,) = n_1 + i_{\pm}\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} - r(A), \tag{4.53}$$

$$\min_{X_1\in\mathcal{S}_1} i_{\pm}(\,X_1\,) = r[\,A_2,\,B_1\,] - i_{\mp}\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix}, \tag{4.54}$$

and

$$\max_{X_3\in\mathcal{S}_3} r(\,X_3\,) = n_2 + r[\,A_1,\,B_2\,] - r(A), \tag{4.55}$$

$$\min_{X_3\in\mathcal{S}_3} r(\,X_3\,) = 2r[\,A_1,\,B_2\,] - r\begin{bmatrix} AB^* & A_1 \\ A_1^* & 0 \end{bmatrix}, \tag{4.56}$$

$$\max_{X_3\in\mathcal{S}_3} i_{\pm}(\,X_3\,) = n_2 + i_{\pm}\begin{bmatrix} AB^* & A_1 \\ A_1^* & 0 \end{bmatrix} - r(A), \tag{4.57}$$

$$\min_{X_3\in\mathcal{S}_3} i_{\pm}(\,X_3\,) = r[\,A_1,\,B_2\,] - i_{\mp}\begin{bmatrix} AB^* & A_1 \\ A_1^* & 0 \end{bmatrix}. \tag{4.58}$$

Applying Lemma 1.5 to (4.51)–(4.54), we easily obtain the following algebraic properties of the submatrix X_1 in (4.47).

Corollary 4.7 *Assume that matrix equation in* (4.47) *has a Hermitian solution. Then,*

(a) (4.47) *has a Hermitian solution in which* X_1 *is nonsingular if and only if* $r[\,A_2,\,B_1\,] = r(A)$.

(b) X_1 *is nonsingular in all Hermitian solutions of* (4.47) *if and only if* $r\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = 2r[\,A_2,\,B_1\,] - n_1$.

(c) (4.47) *has a Hermitian solution in which* $X_1 > 0$ ($X_1 < 0$) *if and only if*

$$i_{+}\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r(A) \quad \left(i_{-}\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r(A)\right).$$

(d) $X_1 > 0$ $(X_1 < 0)$ *in all Hermitian solutions of* (4.47) *if and only if*

$$i_-\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r[\,A_2,\, B_1\,] - n_1 \quad \left(i_+\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r[\,A_2,\, B_1\,] - n_1\right).$$

(e) (4.47) *has a Hermitian solution in which* $X_1 \geqslant 0$ $(X_1 \leqslant 0)$ *if and only if*

$$i_+\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r[\,A_2,\, B_1\,] \quad \left(i_-\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r[\,A_2,\, B_1\,]\right).$$

(f) $X_1 \geqslant 0$ $(X_1 \leqslant 0)$ *in all Hermitian solutions of* (4.47) *if and only if*

$$i_-\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r(A) - n_1 \qquad \left(i_+\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r(A) - n_1\right).$$

(g) (4.47) *has a Hermitian solution in which* $X_1 = 0$ *if and only if* $\mathscr{R}(B_1) \subseteq \mathscr{R}(A_2)$.

(h) $X_1 = 0$ *in all Hermitian solutions of* (4.47) *if and only if* $r[\,A_2,\, B_1\,] = r(A) - n_1$.

(i) *The rank of* X_1 *in the Hermitian solution of* (4.47) *is invariant* $\Leftrightarrow$ *the positive index of inertia of* X_1 *in the Hermitian solution of* (4.47) *is invariant* $\Leftrightarrow$ *the negative index of inertia of* X_1 *in the Hermitian solution of* (4.47) *is invariant* $\Leftrightarrow$ $r\begin{bmatrix} AB^* & A_2 \\ A_2^* & 0 \end{bmatrix} = r[\,A_2,\, B_1\,] + r(A) - n_1.$

5 Conclusions

In this paper, we studied the problems of maximizing and minimizing the rank and inertia of the constrained matrix expression in (1.2) and (1.3), and obtained many symbolic formulas for calculating the extremal ranks and inertias of (1.2) and (1.3) by using pure algebraic operations of matrices and their generalized inverses. As direct applications, we gave necessary and sufficient conditions for the existence of X satisfying the triple matrix equations in (1.2) and (1.3), as well as some matrix inequalities. Although the problems of maximizing and minimizing ranks and inertias of matrices are generally regarded as NP-hard, the results presented in this previous sections as well as the papers [13, 15, 16, 17, 18, 30, 31, 32, 36, 38] show that many closed-form formulas for calculating global extremal ranks inertias of some simpler matrix expressions can be established symbolically by using some pure algebraic operations of matrices, while these explicit formulas can be used to solve many fundamental problems in matrix theory, as mentioned in the beginning of this paper. All the results obtained in these papers are brand-new, but easy to understand within the scope of elementary linear algebra. This series of fruitful researches show that for many basic or classic problems like solvability of matrix equations and matrix inequalities, we are still able to establish a variety of innovative results by some new methods.

Motivated by the fruitful results and the analytical methods used in this paper, we mention some research problems for further consideration:

(a) A challenging task is to give the closed-form for the general common solution of $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ that satisfies $X > 0$ $(< 0, \geqslant 0, \leqslant 0)$, which is equivalent to solving the inequalities

$$X_0 + VF_B + F_BV^* + F_{B_2}UF_{B_3} + F_{B_3}U^*F_{B_2} > 0\,(< 0,\ \geqslant 0,\ \leqslant 0). \tag{5.1}$$

Moreover, give the extremal rank and partial inertia of $A_1 - B_1XB_1^*$ subject to $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ and $\geqslant 0$.

(b) Give the extremal ranks and inertias of the LHMF $A_1 - B_1XB_1^*$ subject to the common Hermitian solution of the $k-1$ consistent linear matrix equations

$$[\,B_2XB_2^*, \ldots, B_kXB_k^*\,] = [\,A_2, \ldots, A_k\,],$$

and to establish necessary and sufficient condition for the set of matrix equations

$$[\,B_1XB_1^*, \ldots, B_kXB_k^*\,] = [\,A_1, \ldots, A_k\,]$$

to have a common Hermitian solution, as well as a common nonnegative definite solution.

(c) Give the extremal rank and partial inertia of $A_1 - B_1XB_1^*$ subject to a linear matrix inequality $B_2XB_2^* \geqslant A_2$. In such a case, it is necessary to first give analytical expression for the general Hermitian solution of $B_2XB_2^* \geqslant A_2$.

(d) Give the extremal ranks and inertias of $A_1 - B_1XB_1^*$ subject to $B_2X = A_2$ and $X \geqslant 0$.

Since linear algebra is a successful theory with essential applications in most scientific fields, the methods and results in matrix theory are prototypes of many concepts and content in other advanced branches of mathematics. In particular, matrix equations and matrix inequalities in the Löwner partial ordering, as well as generalized inverses of matrices were sufficiently extended to their counterparts for operators in a Hilbert space, or elements in a ring with involution, and their algebraic properties were extensively studied in the literature. In most cases, the conclusions on the complex matrices and their counterparts in general algebraic settings are analogous. Also, note that the results in this paper are derived from ordinary algebraic operations of the given matrices and their generalized inverses. Hence, it is no doubt that most of the conclusions in this paper can trivially be extended to the corresponding equations and inequalities for linear operators on a Hilbert space or elements in a ring with involution.

References

[1] E. Candes and B. Recht, Exact matrix completion via convex optimization, Found. of Comput. Math. 9(2009), 717–772.

[2] D. Chu, Y.S. Hung, H.J. Woerdeman, Inertia and rank characterizations of some matrix expressions, SIAM. J. Matrix Anal. Appl. 31(2009), 1187–1226.

[3] M. Fazel, H. Hindi and S. Boyd, A Rank minimization heuristic with application to minimum order system approximation, In: Proceedings of the 2001 American Control Conference, pp. 4734–4739, 2001.

[4] M. Fazel, H. Hindi and S. Boyd, Rank minimization and applications in system theory, In: Proceedings of the 2004 American Control Conference pp. 3273–3278, 2004.

[5] J.F. Geelen, Maximum rank matrix completion, Linear Algebra Appl. 288(1999), 211–217.

[6] J. Groß, A note on the general Hermitian solution to $AXA^* = B$, Bull. Malays. Math. Soc. (2) 21(1998), 57–62.

[7] N.J.A. Harvey, D.R. Karger and S. Yekhanin, The complexity of matrix completion, In: Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithm, Association for Computing Machinery, New York, pp. 1103–1111, 2006.

[8] T.M. Hoang and T. Thierauf, The complexity of the inertia, Lecture Notes in Computer Science 2556, Springer, pp. 206–217, 2002.

[9] T.M. Hoang and T. Thierauf, The complexity of the inertia and some closure properties of GapL, In: Proceedings of the Twentieth Annual IEEE Conference on Computational Complexity, pp. 28–37, 2005.

[10] C.G. Khatri and S.K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math. 31(1976), 579–585.

[11] Y. Kim and M. Mesbahi, On the rank minimization problem, In: Proceedings of the 2004 American Control Conference, Boston, pp. 2015–2020, 2004.

[12] M. Laurent, Matrix completion problems, In: Encyclopedia of Optimization (C.A. Floudas and P.M. Pardalos, eds.), Vol. III, Kluwer, pp. 221–229, 2001.

[13] Ying Li, F. Zhang, W. Guo and J. Zhao, Solutions with special structure to the linear matrix equation $AX = B$, Comput. Math. Appl. 61(2011), 374–383.

[14] X. Liu and J. Rong, On Hermitian nonnegative-definite solutions to matrix equations, Math. Notes 85(2009), 453–457.

[15] Y. Liu and Y. Tian, More on extremal ranks of the matrix expressions $A - BX \pm X^*B^*$ with statistical applications, Numer. Linear Algebra Appl. 15(2008), 307–325.

[16] Y. Liu and Y. Tian, Extremal ranks of submatrices in an Hermitian solution to the matrix equation $AXA^* = B$ with applications, J. Appl. Math. Comput. 32(2010), 289–301.

[17] Y. Liu and Y. Tian, A simultaneous decomposition of a matrix triplet with applications, Numer. Linear Algebra Appl. 18(2011), 69–85.

[18] Y. Liu and Y. Tian, Max-min problems on the ranks and inertias of the matrix expressions $A - BXC \pm (BXC)^*$ with applications, J. Optim. Theory Appl. 148(2011), 593–622.

[19] Y. Liu, Y. Tian, Hermitian-type of singular value decomposition for a pair of matrices and its applications, Numer. Linear Algebra Appl., accepted.

[20] Y. Liu, Y. Tian, Y. Takane, Ranks of Hermitian and skew-Hermitian solutions to the matrix equation $AXA^* = B$, Linear Algebra Appl. 431(2009), 2359–2372.

[21] G. Marsaglia and G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2(1974), 269–292.

[22] M. Mahajan and J. Sarma, On the complexity of matrix rank and rigidity, Lecture Notes in Computer Science Vol. 4649, Springer, pp. 269–280, 2007.

[23] M. Mesbahi, On the rank minimization problem and its control applications, Systems & Control Letters 33(1998), 31–36.

[24] M. Mesbahi and G.P. Papavassilopoulos, Solving a class of rank minimization problems via semi-definite programs, with applications to the fixed order output feedback synthesis, In: Proceedings of the American Control Conference, Albuquerque, New Mexico, pp. 77–80, 1997.

[25] B.K. Natarajan, Sparse approximate solutions to linear systems, SIAM J. Comput. 24(1995), 227–234.

[26] A B. Özgüler and N. Akar, A common solution to a pair of linear matrix equations over a principal ideal domain, Linear Algebra Appl. 144(1991), 85–99.

[27] B. Recht, M. Fazel and P.A. Parrilo, Guaranteed minimum rank solutions to linear matrix equations via nuclear norm minimization, SIAM Review 52(2010), 471–501.

[28] G.A.F. Seber, A Matrix Handbook for Statisticians, John Wiley & Sons, 2008.

[29] Y. Tian, Solvability of two linear matrix equations, Linear Multilinear Algebra 48(2000), 123–147.

[30] Y. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, Linear Algebra Appl. 433(2010), 263–296.

[31] Y. Tian, Rank and inertia of submatrices of the Moore–Penrose inverse of a Hermitian matrix, Electron. J. Linear Algebra 20(2010), 226–240.

[32] Y. Tian, Completing block Hermitian matrices with maximal and minimal ranks and inertias. Electron. J. Linear Algebra 21(2010), 124–141.

[33] Y. Tian, Maximization and minimization of the rank and inertia of the Hermitian matrix expression $A-BX-(BX)^*$ with applications, Linear Algebra Appl. 434(2011), 2109–2139.

[34] Y. Tian, Solving optimization problems on ranks and inertias of some constrained nonlinear matrix functions via an algebraic linearization method, Nonlinear Anal. 75(2012), 717–734.

[35] Y. Tian, Solutions to 18 constrained optimization problems on the rank and inertia of the linear matrix function $A + BXB^*$, Math. Comput. Modelling 55(2012), 955–968.

[36] Y. Tian, On additive decompositions of the Hermitian solutions of the matrix equation $AXA^* = B$, Mediterr. J. Math. 9(2012), 47–60.

[37] Y. Tian, Least-squares solutions and least-rank solutions of the matrix equation $AXA^* = B$ and their relations, Numer. Linear Algebra Appl., DOI:10.1006/nla.829.

[38] Y. Tian and Y. Liu, Extremal ranks of some symmetric matrix expressions with applications, SIAM J. Matrix Anal. Appl. 28(2006), 890–905.

[39] J. Wang, V. Sreeram and W. Liu, The parametrization of the pencil $A+BKC$ with constant rank and its application, Internat. J. Infom. Sys. Sci. 4(2008), 488–499.

[40] X. Zhang, The general common Hermitian Nonnegative-definite solution to the matrix equations $AXA^* = B$ and $CXC^* = D$, Linear Multilinear Algebra 52(2004), 49–60.

[41] X. Zhang, The general common Hermitian nonnegative-definite solution to the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$ with applications in statistics, J. Multivariate Anal. 93(2005), 257–266.